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Comprehensive Analysis of a Curve with an Asymptote in Polar Coordinates

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Abstract

Objective: *To develop an algorithm for the analysis of functions with asymptotes in the polar coordinate system and apply it to the study of curves.*

Theoretical Background*: The study of curves has broad applications: in physics — for analyzing the trajectories of moving bodies; in astronomy — for examining the orbits of planets and satellites; in engineering — for designing aerodynamic shapes; and in computer graphics — for modeling geometric objects.*

Analyzing curves is an essential tool for understanding complex relationships. Methods of differential and integral calculus, as well as representing curves in various coordinate systems, allow researchers to identify key properties such as extrema, asymptotes, discontinuities, and other features. These approaches are widely applied across numerous fields of science and technology.

Method: *The study employs methods of analysis and synthesis, as well as the mathematical apparatus of differential calculus. Particular attention is given to the investigation of a function in the polar coordinate system using derivatives to identify the key features of its behavior.*

Results and Discussion: *As a result of the study, an algorithm was developed for analyzing functions with asymptotes in the polar coordinate system, including cases involving discontinuities. The proposed method makes it possible to identify the key points of the curve without the use of computational tools, which makes it convenient for manual plotting in both polar and Cartesian coordinate systems. This approach provides a deeper understanding of the behavior of unbounded functions and expands the range of methods available for graphical analysis.*

Research implications: *A systematic approach to the study of curves in the polar coordinate system is presented, including the identification of oblique asymptotes and characteristic points without the use of computational software. This broadens the existing methodologies for function analysis and visualization.*

Originality/Value: *The work contributes to the development of analytical methods for studying curves defined in polar coordinates, offering both methodological and practical significance in engineering, physics, and mathematical disciplines.*

Keywords: *function, derivative, polar coordinates, curves, algorithm, asymptotes.*

1 Introduction

Curves have long played a fundamental role in mathematics, physics, astronomy, mechanics, and engineering. Their study began with basic geometric constructions and gradually evolved to incorporate analytical, algebraic, differential, and numerical methods. Today, curve analysis remains relevant, particularly in the context of advances in computational geometry, mathematical modeling, and engineering applications.

The systematic study of curves dates back to ancient Greek mathematics. Euclid and Archimedes made significant contributions to the development of geometry, including the investigation of circles and more complex curves such as the Archimedean spiral [1].

The analysis of curves also has important practical applications in various technical fields. In particular, polar coordinate representations are used in radar systems to describe spatial relationships involving range and azimuth, demonstrating the practical relevance of polar-coordinate methods in radar-related applications [2]. This provides an additional motivation for studying the geometric properties of curves defined in polar coordinates.

Historically, methods for studying curves have evolved considerably. The main stages of this development can be summarized as follows:

- geometric methods (Ancient Greece): constructing and investigating curves using geometric techniques;
- algebraic methods: representing curves by equations;
- analytical methods: establishing relationships between algebra and geometry;
- differential methods: investigating tangents, extrema, curvature, convexity, and concavity;
- methods of differential geometry and topology: analyzing more complex geometric structures;
- computational methods: modeling and visualizing curves using numerical algorithms and computer-based techniques.

Modern curve analysis integrates analytical, differential, and numerical methods, making it possible to investigate substantially more complex mathematical objects and their properties.

The study of functions is most commonly carried out within the Cartesian coordinate system. Classical function analysis typically includes determining the domain, identifying points of intersection with the coordinate axes, analyzing monotonicity and extrema, examining convexity and concavity, locating inflection points, determining asymptotic behavior, and constructing the graph of the function.

However, this approach is not always convenient, since the equation of a curve in Cartesian coordinates may not be representable in explicit form, making traditional analysis difficult. In contrast, certain curves can be represented more naturally by functions in the polar coordinate system, which provides alternative possibilities for their analytical investigation.

Therefore, the development of a systematic approach to the analysis of curves defined in polar coordinates is of both theoretical and practical interest. This paper presents an algorithm for the comprehensive analysis of functions defined in polar coordinates, with particular attention to extrema, convexity and concavity, and asymptotic behavior.

2 Analysis of Recent Studies and Publications

The study of curves and their geometric properties is one of the classical areas of mathematical analysis and geometry. Over time, methods for investigating curves have developed from purely geometric constructions to analytical and differential approaches. The introduction of coordinate systems and differential calculus made it possible to study the behavior of curves quantitatively, including their extrema, convexity, concavity, and asymptotic behavior.

Classical approaches to curve analysis are primarily associated with the Cartesian coordinate system. However, certain curves have equations that cannot be conveniently represented as explicit functions in Cartesian coordinates. In such cases, the polar coordinate system provides an alternative representation in which a curve may be expressed as a radius function depending on an angular variable. This makes it possible to apply methods of differential calculus directly to the investigation of the curve.

Several previous studies by Nauryzbayeva have addressed individual aspects of the analysis of curves in the polar coordinate system. One of the earlier studies examined the conditions of convexity and concavity of a curve at the extremum points of the radius function [4]. The study proposed an approach for determining the local geometric behavior of a polar curve at its characteristic points. These conditions provide a theoretical basis for distinguishing convex and concave behavior at the extrema of the radius function and are applied in the present research as one of the stages of comprehensive curve analysis. The 2011 work is also cited as a theoretical basis in the author's later study on extrema.

Another important aspect of curve analysis is the investigation of its behavior at infinity. Nauryzbayeva [3] considered the problem of determining oblique asymptotes of functions defined in the polar coordinate system. The study demonstrated that when a curve can be represented explicitly as a function in polar coordinates, its asymptotic behavior can be investigated analytically. Particular attention was given to the determination of oblique asymptotes and to the specific features that distinguish their analysis in polar coordinates from the corresponding procedures in the Cartesian coordinate system.

The problem of determining extremum points of curves was further developed by Nauryzbayeva [5]. An algorithm was proposed for finding the extrema of certain curves represented in the polar coordinate system and for analyzing them in both polar and Cartesian coordinates. The study demonstrated that the use of the radius function and its derivatives makes it possible to identify characteristic points of a curve and subsequently determine their corresponding positions in the Cartesian coordinate system.

These previous studies addressed important components of curve analysis separately: convexity and concavity at the extrema of the radius function [4], determination of oblique asymptotes [3], and identification of extremum points [5]. However, a comprehensive analysis of a curve requires these individual procedures to be integrated into a unified sequence.

Therefore, the present study develops the previous research by combining these approaches within a single algorithm for the comprehensive analysis of a curve with asymptotes in the polar coordinate system. The proposed algorithm includes determining the domain of the radius function, identifying extremum points, examining convexity and concavity, transforming characteristic points into Cartesian coordinates, determining extrema with respect to the Cartesian axes, investigating different types of asymptotes, and constructing the final graph.

Thus, the novelty of the present study lies in the integration of previously developed analytical approaches into a systematic procedure that enables the principal geometric characteristics of a curve to be investigated consistently in both polar and Cartesian coordinate systems. This provides a more complete analytical representation of the curve and expands the methodological possibilities for its investigation without exclusive reliance on computational software.

3 Theoretical Foundations

To develop a comprehensive understanding of curve analysis in Cartesian and polar coordinate systems, it is first necessary to distinguish between the types of polar coordinate systems.

Two primary forms of polar coordinates are commonly recognized:

- strict polar coordinates, where $\rho \geq 0$;

generalized polar coordinates, where ρ may take both positive and negative values, i.e., $\rho \geq 0$ or $\rho \leq 0$.

In the present study, we focus on the first type—strict polar coordinates. The identification of asymptotes represents one of the key stages in the analysis of functions. Unlike in the Cartesian coordinate system, where asymptotes typically follow well-defined linear or curvilinear forms, functions expressed in polar coordinates can exhibit four distinct types of asymptotic behavior.

Reference [3] provides a classification of the principal types of asymptotes in the polar coordinate system, together with analytical formulas for their determination.

1) Oblique Asymptote

In the Cartesian and polar coordinate systems, an oblique asymptote can be represented by the following equations, respectively:

$$y = kx + b \text{ and } \quad \rho = \frac{b}{\sin\phi - k\cos\phi} \tag{1}$$

In the polar coordinate system, an oblique asymptote should be identified at points where one of the limits $\lim\limits_{\phi\to\phi_0-0}\rho$, $\lim\limits_{\phi\to\phi_0+0}\rho$ tends to infinity, and the coefficients are determined using the following formula:

$$k = \lim_{\phi\to\phi_0} tg\phi,\ \ b = \lim_{\phi\to\phi_0}(\rho\sin\phi - k\rho\cos\phi). \tag{2}$$

2) Horizontal Asymptote

It is identified at points where one of the limits $\lim\limits_{\phi\to 0-}\rho$, $\lim\limits_{\phi\to 0+}\rho$, $\lim\limits_{\phi\to \pi-0}\rho$, $\lim\limits_{\phi\to \pi+0}\rho$ tends to infinity. Under these conditions, $k = \lim\limits_{\phi\to\phi_0} tg\phi = 0$. If one of the limits $\lim\limits_{\phi\to 0+}\rho\sin\phi$, $\lim\limits_{\phi\to 0-}\rho\sin\phi$, $\lim\limits_{\phi\to \pi+}\rho\sin\phi$, $\lim\limits_{\phi\to \pi-}\rho\sin\phi$ is a finite number and equals b, then the equation of the horizontal asymptote is:

$$\rho = \frac{\mathrm{b}}{\sin\phi}. \tag{3}$$

In the polar coordinate system, a horizontal asymptote does not restrict the behavior of the radial function as it approaches infinity.

Example. The function $\rho = \frac{\sin\phi - 0,5}{\sin\phi}$ has a horizontal asymptote $\rho = \frac{-0,5}{\sin\phi}$.

3) Vertical Asymptote

It is identified at points where one of the limits $\lim_{\phi\to\frac{\pi}{2}-0}\rho$, $\lim_{\phi\to\frac{\pi}{2}+0}\rho$, $\lim_{\phi\to\frac{3\pi}{2}-0}\rho$, $\lim_{\phi\to\frac{3\pi}{2}+0}\rho$ tends to infinity. If at least one of the limits $\lim_{\phi\to\frac{\pi}{2}-0}\rho\cos\phi$, $\lim_{\phi\to\frac{\pi}{2}+0}\rho\cos\phi$, $\lim_{\phi\to\frac{3\pi}{2}-0}\rho\cos\phi$, $\lim_{\phi\to\frac{3\pi}{2}+0}\rho\cos\phi$ tends to a constant value b, then the line

$$\rho = \frac{b}{\cos\phi} \tag{4}$$

is a vertical asymptote.

Example. The function $\rho = \frac{\sin\phi + 1}{\cos\phi}$ has a vertical asymptote $\rho = \frac{1}{\cos\phi}$.

4) *Circular Asymptote*

If for a given curve the limit $\lim_{\phi\to\infty}\rho$ is a finite number r, then the equation $\rho = r$ represents a circular asymptote of the curve.

Example. The function $\rho = \frac{\phi + 1}{\phi}$ has a circular asymptote $\rho = 1$

If there exists an equation of the curve $\rho = \rho(\phi)$ in the polar coordinate system, it can be considered as a function of a single variable φ, where $0 \le \phi < \infty$.

In [4], a definition of convexity and concavity at the extreme points is provided.

Definition. The graph of a function in the polar coordinate system is said to be convex at the extreme points of the function's radius if the normal at those points is directed toward the pole, and concave if the normal is directed away from the pole.

A sufficient condition for the graph of the function to be concave or convex at the extreme points is:

If

$$\rho - \rho'' > 0 \tag{5}$$

then the curve is convex; and if

$$\rho - \rho'' < 0 \tag{6}$$

then the curve is concave with respect to the pole.

An algorithm for locating the extreme points of curves is discussed in [4].

4 Methodology

To develop an algorithm for the comprehensive analysis of a function defined in the polar coordinate system, we consider the following function and construct its graph.

$$(x^2 - y^2)^2 = 4y^4 - 4y^3 - 4y^2 - 4yx^2. \tag{7}$$

Solution: First, we derive the equation of the curve in the polar coordinate system. After applying the necessary transformations, it takes the following form.

$$\rho^2(\cos^2 2\phi - 4\sin^4\phi) + 4\rho\sin\phi + 4\sin^2\phi = 0 \tag{8}$$

Let us solve equation (8) with respect to ρ:

$$\rho_{1/2} = \frac{-4\sin\phi \pm \sqrt{16\sin^2\phi - 16\sin^2\phi\,(\cos^2 2\phi - 4\sin^4\phi)}}{2(\cos^2 2\phi - 4\sin^4\phi)} =$$

$$= \frac{-4\sin\phi \pm |4\sin\phi|\sqrt{1-(\cos^4\phi-\sin^4\phi)+2\sin^2\phi(\cos^2\phi+\sin^2\phi)}}{2[(\cos^4\phi-\sin^4\phi)-2\sin^2\phi(\cos^2\phi+\sin^2\phi)]} = \frac{-4\sin\phi\pm 8\sin^2\phi}{2(1-4\sin^2\phi)}. \qquad (9)$$

It follows from equation (9) that:

$$\rho_1 = \frac{\sin\phi}{\sin\phi-0{,}5} \quad \text{and} \quad \rho_2 = \frac{-\sin\phi}{\sin\phi+0{,}5} \qquad (10)$$

It is necessary to analyze the resulting equations (10) of the two curves in the polar coordinate system.

We begin by examining the equation of the first curve. To determine its domain, we equate both the numerator and the denominator of the fraction in the equation to zero, plot the obtained solutions on the trigonometric circle, and identify the signs of the numerator and denominator within the corresponding intervals (Figure 1).

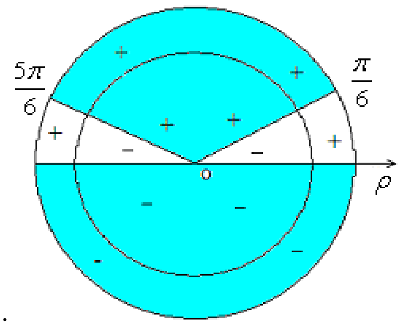

Figure 1 – Domain of the curve: $\rho = \frac{\sin\phi}{\sin\phi-0{,}5}$

Then, the domain of the curve $\rho = \frac{\sin\phi}{\sin\phi-0{,}5}$ is $(\pi/6,\ 5\pi/6) \cup [\pi, 2\pi]$. On the interval $[\pi, 2\pi]$, the equation of the curve is continuous since $\sin\phi - 0{,}5 \neq 0$, and a petal of the curve lies within this segment.

To find the extremal points of the petal, we take the derivative of the radial function and set it equal to zero.

$$\rho_1' = \left(\frac{\sin\phi}{\sin\phi-0{,}5}\right)' = \frac{\cos\phi(\sin\phi-0{,}5)-\sin\phi\cos\phi}{(\sin\phi-0{,}5)^2} = \frac{-0{,}5\cos\phi}{(\sin\phi-0{,}5)^2} = 0 \qquad (11)$$

Solving equation (11), we obtain the solution.

$$-0{,}5\cos\phi = 0 \Rightarrow \phi_1 = \pi/2, \phi_2 = 3\pi/2. \qquad (12)$$

To verify the sufficient conditions for convexity and concavity given in equations (5) and (6) in the polar coordinate system, we compute the second derivative.

$$\rho_1'' = \left(\frac{-0{,}5\cos\phi}{(\sin\phi-0{,}5)^2}\right)' = \frac{0{,}5\sin^2\phi-0{,}25\sin\phi+\cos^2\phi}{(\sin\phi-0{,}5)^2} \qquad (13)$$

Using (10) and (13), we observe that the radius function ρ_1 on the interval $[\pi; 2\pi]$ attains an extremum at $\phi_1 = 3\pi/2$, and is convex, as the sufficient condition for convexity in the polar coordinate system is satisfied.

$$\rho_1(3\pi/2) - \rho_1''(3\pi/2) = 2 - \frac{1}{(-1.5)^3} > 0 \quad 0, \Rightarrow \phi_1 = 3\pi/2 - max. \qquad (14)$$

The point corresponding to the center of the petal has polar coordinates $A\left(\frac{3\pi}{2}; -\frac{2}{3}\right)$.

Its coordinates in the Cartesian coordinate system are: $A\left(\frac{3\pi}{2}; -\frac{2}{3}\right)$

$$x = \frac{\sin\phi}{\sin\phi-0{,}5}\cos\phi\Big|_{\phi=\frac{3\pi}{2}} = 0;\ y = \frac{\sin\phi}{\sin\phi-0{,}5}\sin\phi\Big|_{\phi=\frac{3\pi}{2}} = -\frac{2}{3}. \qquad (15)$$

According to equation (15), the endpoint of the petal axis has the Cartesian coordinates: $A\left(0; -\frac{2}{3}\right)$.

To find the extremum of the curve in the Cartesian coordinate system, its equation is considered as a function $y = f(\phi)$. In this case, the ordinate of each point on the curve is determined by the given function.

$$h_1 = \rho sin\phi = \frac{sin\,\phi}{sin\,\phi - 0{,}5} sin\,\phi = \frac{sin^2\,\phi}{sin\,\phi - 0{,}5}. \tag{16}$$

Differentiating function (16), we obtain the first derivative:

$$h_1' = \frac{2\,sin\,\phi\,cos\,\phi(sin\,\phi - 0{,}5) - cos\,\phi\,sin^2\,\phi}{(sin\,\phi - 0{,}5)^2} = \frac{sin\,\phi\,cos\,\phi(sin\,\phi - 1)}{(sin\,\phi - 0{,}5)^2}; \tag{17}$$

Solving the equation $sin\,\phi\,cos\,\phi\,(sin\,\phi - 1) = 0$, we determine the critical points. After solving the equation, we select only those solutions that lie within the domain of the function. The sufficient condition for an extremum can then be easily verified by analyzing the sign of the first derivative in the neighborhood of each critical point. A maximum is attained at the point (0, 0), while minima occur at the points $(\pi/2; 2)$ and $(3\pi/2; -2/3)$.

In the Cartesian coordinate system, these points correspond to:

$$A\left(0; -\frac{2}{3}\right), O(0;0), B(0;2) \tag{18}$$

The points A and O lie along the central axis of the «petal». The previously identified extrema are defined with respect to the $x-$axis.

Now, let us determine the extreme points of the curve by treating its implicit equation as a function of x with respect to y.

In this case, the x −coordinate of each point on the curve is defined by the given function.

$$h_2 = \rho cos\phi = \frac{sin\,\phi cos\phi}{sin\,\phi - 0{,}5}. \tag{19}$$

Differentiating function (19), we obtain:

$$h_2' = \left(\frac{sin\,\phi\,cos\,\phi}{sin\,\phi - 0{,}5}\right)' = \left(\frac{0{,}5\,sin\,2\phi}{sin\,\phi - 0{,}5}\right)' = \frac{-\,sin^3\,\phi + sin^2\,\phi - 0{,}5}{(sin\,\phi - 0{,}5)^2}; \tag{20}$$

Equating the derivative (20) to zero, we obtain the equation:

$$\frac{-\,sin^3\,\phi + sin^2\,\phi - 0{,}5}{(sin\,\phi - 0{,}5)^2} = 0 \tag{21}$$

In equation (21), let $sin\,\phi = z$, then we have:

$$z^3 - z^2 + 0{,}5 = 0; \tag{22}$$

To solve equation (22) using the interval method, we consider three cases.

Case I. Let us examine the equation $z^3 = z^2 - 0{,}5$ and represent the sign changes of the left- and right-hand sides on the number line (see Fig. 2).

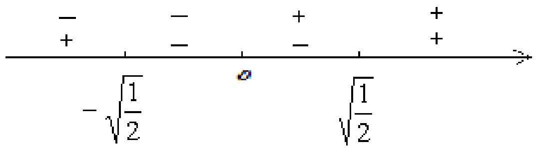

Fig. 2 – Signs of the left- and right-hand sides of the equation $z^3 = z^2 - 0{,}5$

The intervals over which both sides of the equation share the same sign are as follows:

$$z \in \left[-\sqrt{\frac{1}{2}}, 0\right] \cup \left[\sqrt{\frac{1}{2}}, +\infty\right)$$

II. The sign changes of the left and right sides of the equation $z^3 - z^2 = -0{,}5$ are shown in Figure 3.

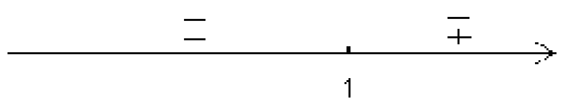

Fig. 3 – Signs of the left- and right-hand sides of the equation $z^3 - z^2 = -0{,}5$

We are interested in the interval $z \in (-\infty; 1]$.

III. The sign changes of the left-hand side and right-hand side of the equation $2z^3 + 1 = 2z^2$are illustrated in Figure 4.

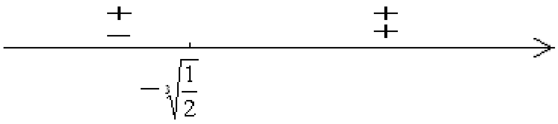

Fig. 4 – Signs of the left and right sides of the equation $2z^3 + 1 = 2z^2$

The interval on which both sides of the equation have the same sign: $z \in \left[-\sqrt[3]{\frac{1}{2}}; +\infty\right)$.

By combining the three considered cases, we find the intervals in which the roots of the original equation are located:

$$z \in \left[-\sqrt{\frac{1}{2}}; 0\right] \cup \left[\sqrt{\frac{1}{2}}; 1\right].$$

In the second case, for the equation $z^3 - z^2 = -0{,}5$ the boundary points of the interval $\left[-\sqrt{\frac{1}{2}}; 0\right]$ are not zeros of the given equation. Let us denote $q_1 = z^3 - z^2$ and $q_2 = -0{,}5$, and calculate their values near the ends of the interval: $q_1(-0{,}707) = -0{,}85$; $q_2(-0{,}707) = -0{,}5$. Then $\Delta(-0{,}707) = -0{,}35 < 0$ and $q_1(-0{,}1) = -0;011, q_2(-0{,}1) = -0{,}5$; $\Delta(-0{,}1) = 0{,}489 > 0, so\ \Delta(-0{,}1) = 0{,}489 > 0$.

The values of Δ at the endpoints of the interval have opposite signs; therefore, a root exists within this interval. Moreover, the root is located closer to the left endpoint, since the absolute value of $\Delta(-0.707)$ is smaller.

Then $q_1(-0{,}6) = -0{,}576; q_2(-0{,}6) = -0{,}5$; Δ(–0,6)= – 0;076<0.

At the point $z = -0{,}5$, the value of $q_1(-0{,}5) = -0{,}375; q_2(-0{,}5) = -0{,}5$; Δ(–0,5)=0,125>0.

To improve the accuracy of the root, we take its value in the [–0,6;–0,56] as $z = -0{,}565$. Then: $q_1(-0{,}565) = -0{,}499587$; $q_2(-0{,}565) = -0{,}0005$; $\Delta(-0{,}565) = 0{,}0004$.

Therefore, $z = -0{,}565$ is a root of the equation.

On the interval $\left[\sqrt{\frac{1}{2}}; 1\right]$, the equation under consideration has no roots, since within this interval the sign of Δ remains unchanged in Cases II and III.

Thus, the obtained root is accurate to three decimal places.

Since $\sin\phi = z$, then:

$$\phi = (-1)^k \arcsin(-0{,}565) + k\pi \Rightarrow \phi = (-1)^{k+1} \arcsin(0{,}565) + k\pi \quad (23)$$

We need the solutions that belong to the interval [π, 2π].

$$\phi_1 = -\arcsin 0{,}565 \text{ and } \phi_2 = \arcsin 0{,}565 + \pi. \quad (24)$$

Now, let us determine the values of the radius function corresponding to these points.

We will verify the sufficient condition for an extremum in the Cartesian coordinate system.

To do so, we compute the second derivative of function (19).

$$h_2'' = \left(\frac{\sin\phi\cos\phi}{\sin\phi - 0{,}5}\right)' = \left(\frac{-\sin^3\phi - 0{,}5\cos 2\phi}{(\sin\phi - 0{,}5)^2}\right)' = \frac{\cos\phi(-\sin^3\phi + 1{,}5\sin\phi - \sin\phi + 1)}{(\sin\phi - 0{,}5)^3}. \quad (25)$$

Taking into account that sinϕ $sin\,\phi = -0{,}565$ and $cos\,\phi = \sqrt{1-\sin^2\phi} = \sqrt{1-0{,}565^2} = 0{,}8251$ and equation (25), we have:

$$h_2''(-arcsin\,0{,}565) = \frac{0{,}8251\cdot(-(-0{,}565)^3+1{,}5\cdot(-0{,}565)^2+0{,}565+1)}{(-0{,}565-0{,}5)^3} < 0; \tag{26}$$

$$h_2''(arcsin\,0{,}565+\pi) = \frac{-0{,}8251\cdot(-(-0{,}565)^3+1{,}5\cdot(-0{,}565)^2+0{,}565+1)}{(-0{,}565-0{,}5)^3} > 0. \tag{27}$$

After verifying the sufficient conditions given in (26) and (27), we conclude that

$$\phi_1 = -\,arcsin\,0{,}565 - max;\quad \phi_2 = arcsin\,0{,}565 + \pi - min. \tag{28}$$

According to equation (23), the value of the radius function is given by:

$$\rho = \frac{sin(-arcsin\,0{,}565)}{sin(-arcsin\,0{,}565)-0{,}5} = \frac{-0565}{-0{,}565-0{,}5} = 0{,}5305. \tag{29}$$

And for $\phi_2 = arcsin\,0{,}565+\pi$, the value of the radius function is $\rho = 0{,}5305$, since this angle lies in the third quadrant.

Now, let us find the points in the Cartesian coordinate system that correspond to these values.

$$x = \rho\,cos\,\phi = 0{,}5305(\sqrt{1-0{,}565^2}) \approx 0{,}4;\;\; y = \rho\,sin(-\,arcsin\,0{,}565) \approx -0{,}3. \tag{30}$$

When determining x, we take the positive root, since the angle $\phi = -\,arcsin\,0{,}565$ lies in the fourth quadrant. According to equation (28), we find the coordinates of the extremum points in the Cartesian coordinate system: $C(0{,}4;-0{,}3)$ and $D(-0{,}4;-0{,}3)$.

At the third stage of the analysis, we will calculate the asymptotes of the curve. They should be sought among the values of ϕ for which $\rho \to \infty$.

This condition is satisfied at the boundary points of the intervals [0;π/6)∪(5π/6;2π], that is, when $\phi \to \pi/6$, $\phi \to 5\,\pi/6$, the radius function $\rho \to \infty$.

Oblique asymptotes may exist at these values.

Using formulas (1) and (2), as well as L'Hôpital's Rule, we will proceed with further calculations.

$$k_1 = \lim_{\phi\to\frac{\pi}{6}} tg\phi = \frac{\sqrt{3}}{3}, \tag{31}$$

$$b_1 = \lim_{\phi\to\frac{\pi}{6}} \rho(sin\phi - k_1 cos\phi) = \lim_{\phi\to\frac{\pi}{6}} \frac{sin\phi(sin\phi - \frac{\sqrt{3}}{3}cos\phi)}{sin\phi - 0{,}5} =$$

$$= \lim_{\phi\to\frac{\pi}{6}} \frac{cos\phi\left(sin\phi-\frac{\sqrt{3}}{3}cos\phi\right)+sin\phi(cos\phi+\frac{\sqrt{3}}{3}sin\phi)}{cos\phi} = \frac{2}{3} \tag{32}$$

$$k_2 = \lim_{\phi\to\frac{5\pi}{6}} tg\phi = -\frac{\sqrt{3}}{3}, \tag{33}$$

$$b_2 = \lim_{\phi\to\frac{5\pi}{6}} \rho(sin\phi - k_2 cos\phi) = \lim_{\phi\to\frac{5\pi}{6}} \frac{sin\phi(sin\phi + \frac{\sqrt{3}}{3}cos\phi)}{sin\phi - 0{,}5} =$$

$$= \lim_{\phi\to\frac{5\pi}{6}} \frac{cos\phi\left(sin\phi+\frac{\sqrt{3}}{3}cos\phi\right)+sin\phi(cos\phi-\frac{\sqrt{3}}{3}sin\phi)}{cos\phi} = \frac{2}{3}. \tag{34}$$

Therefore, the asymptotes, according to equations (31)–(34), are as follows.

$$y = \frac{\sqrt{3}}{3}x + \frac{2}{3},\qquad y = -\frac{\sqrt{3}}{3}x + \frac{2}{3}. \tag{35}$$

Let us plot the graph, displaying these asymptotes along with the points at the ends of the petal's axis of symmetry: $A(0;-2/3), O(0;0)$; the minimum with respect to the x-axis at point $B(0;2)$; and the extremum points with respect to the y-axis: $C(0{,}4;-0{,}3)$, $D(-0{,}4;-0{,}3)$ (Fig. 5).

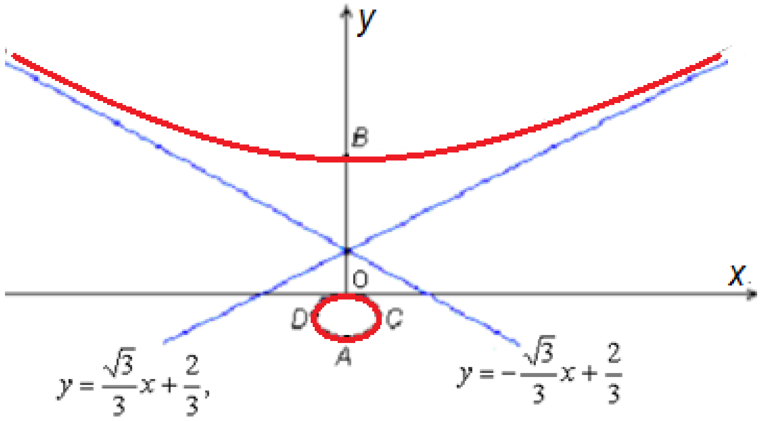

Figure 5 – Graph of the function $\rho = \frac{\sin\phi}{\sin\phi - 0{,}5}$

Let us consider the second curve: $\rho = \frac{-\sin\phi}{\sin\phi+0{,}5}$.

The domain of the curve: $\left[\pi;\frac{7\pi}{6}\right) \cup \left(\frac{11\pi}{6};\ 2\pi\right]$ (Fig. 6).

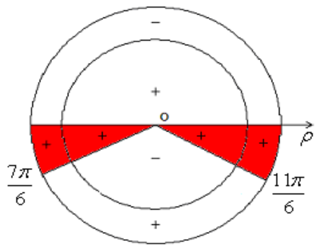

Figure 6 – The Domain of the Function $\rho = \frac{-\sin\phi}{\sin\phi+0{,}5}$

At the boundaries of this interval, the radius function is discontinuous. Therefore, in this case, there will be no “petal.” If there are any extremum points, we will determine them.

We will search for the extrema of the curve using the function $h = \rho\sin\phi$.

$$h_1' = (\rho\sin\phi)' = \left(\frac{-\sin^2\phi}{\sin\phi+0{,}5}\right)' = \frac{-2\sin\phi\cos\phi(\sin\phi+0{,}5)+\cos\phi\sin^2\phi}{(\sin\phi-0{,}5)^2} = \frac{\sin\phi\cos\phi(-\sin\phi-1)}{(\sin\phi-0{,}5)^2}. \quad (36)$$

The derivative (36) is equal to zero when $\sin\phi\cos\phi\,(\sin\phi+1) = 0$.

The solutions of this equation that belong to the domain of the curve $\phi = \pi, \phi = 2\pi$..

At both of these points, the sign of h′ changes from «+» to «–»; therefore, at the points $(\pi;0)$ and $(2\pi;0)$, the curve reaches its maximum.

In the Cartesian coordinate system, these values correspond to the point $O(0;0)$, since $\rho(0) = 0$.

Now, let us find the extremum points of the curve by considering its implicit equation as a function of x in terms of y.

In this case, the x-coordinate of each point on the curve is determined by this function.

$$h_2' = (\rho\cos\phi)' = \left(\frac{-\sin\phi\cos\phi}{\sin\phi+0{,}5}\right)' = \left(\frac{-0{,}5\sin 2\phi}{\sin\phi+0{,}5}\right)' = \frac{\sin^3\phi+\sin^2\phi-0{,}5}{(\sin\phi-0{,}5)^2}; \quad (37)$$

By setting the derivative (37) equal to zero, we obtain the equation.

$$\frac{\sin^3\phi+\sin^2\phi-0{,}5}{(\sin\phi-0{,}5)^2} = 0 \Rightarrow \sin^3\phi+\sin^2\phi-0{,}5 = 0. \quad (38)$$

If we let $\sin\phi = z$ in equation (38), we obtain the equation:

$2z^3 + 2z^2 - 1 = 0$. Using the interval method, we find that the solutions of this equation lie within the interval $z \in \left[-1; -\sqrt{1/2}\right] \cup \left[0; \sqrt{1/2}\right]$.

However, the corresponding values of ϕ do not belong to the interval $\left[\pi; \frac{7\pi}{6}\right) \cup \left(\frac{11\pi}{6}; 2\pi\right]$.

Therefore, this curve has no extrema if x is considered as a function of y.

The curve tends to infinity at $\phi = \frac{7\pi}{6}, \phi = \frac{11\pi}{6}$. Using formulas (1) and (2), we find that its oblique asymptotes are as follows:

$$y = \frac{\sqrt{3}}{3}x + \frac{2}{3}, \qquad y = -\frac{\sqrt{3}}{3}x + \frac{2}{3}$$

Let us plot the graph of the curve, showing this asymptote and the minimum point $O(0;\ 0)$ (Fig. 7).

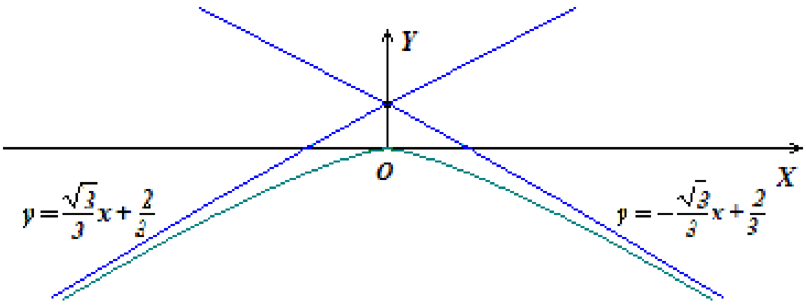

Figure 7 – Graph of the function $\rho = \frac{-\sin\phi}{\sin\phi + 0{,}5}$

Therefore, the graph of the curve originally defined by the implicit equation (1) is as follows (Fig. 8).

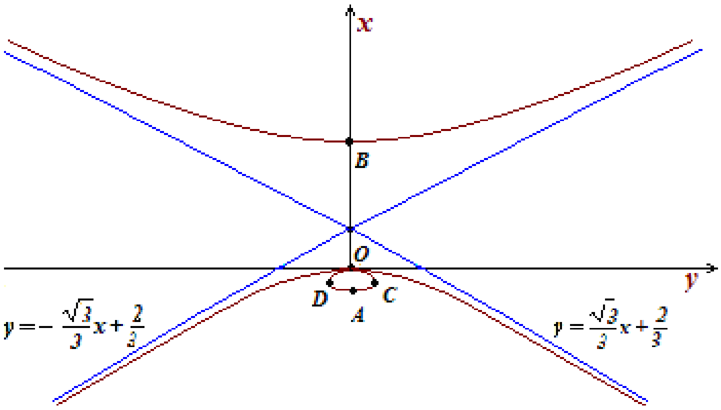

Figure 8 – Graph of the Curve $(x^2 - y^2)^2 = 4y^4 - 4y^3 - 4y^2 - 4yx^2$

5 Research Results

The conducted study resulted in the development of a systematic algorithm for the comprehensive analysis of curves with asymptotes defined in the polar coordinate system. The proposed approach combines the analysis of a curve in polar coordinates with the determination of its corresponding geometric characteristics in the Cartesian coordinate system.

The application of the proposed method to the investigated curve demonstrated that its analysis can be performed sequentially by determining the domain, identifying the critical points of the radius function, examining convexity and concavity at the extremum points, determining the corresponding Cartesian coordinates, and calculating the asymptotes.

Based on the calculations performed in the study, the following algorithm for the comprehensive analysis of a function defined in the polar coordinate system was obtained:

1. Determine the domain of the function.
2. Find the critical points of the radius function and verify the sufficient conditions for convexity and concavity at the extremum points.
3. Calculate the Cartesian coordinates of the extremum points of the radius function.
4. Identify the critical points of the corresponding function in the Cartesian coordinate system and verify the sufficient condition for an extremum.

5. Determine the Cartesian coordinates of the extremum points.
6. Determine the extremum points by considering the implicit equation of the curve with respect to the corresponding Cartesian variable and verify the sufficient condition for an extremum.
7. Identify the oblique asymptotes at the points where the limits $\lim_{\phi \to \phi_0 - 0} \rho$ and $\lim_{\phi \to \phi_0 + 0} \rho$ tend to infinity.
8. Find the horizontal and vertical asymptotes at the points where the one-sided limits $\lim_{\phi \to 0-} \rho$, $\lim_{\phi \to 0+} \rho$, $\lim_{\phi \to \pi - 0} \rho$, $\lim_{\phi \to \pi + 0} \rho$ and $\lim_{\phi \to \frac{\pi}{2} - 0} \rho$, $\lim_{\phi \to \frac{\pi}{2} + 0} \rho$, $\lim_{\phi \to \frac{3\pi}{2} - 0} \rho$, $\lim_{\phi \to \frac{3\pi}{2} + 0} \rho$ tend to infinity.
9. Examine the existence of a circular asymptote by analyzing the limiting behavior of the radius function.
10. Plot the graph of the investigated curve using the obtained characteristic points and asymptotes.

The application of this algorithm to the curve considered in the study made it possible to determine its characteristic points, extrema, asymptotes, and graphical behavior. The calculations performed in both polar and Cartesian coordinate systems provided complementary information about the geometric properties of the curve.

Thus, the results demonstrate that when the equation of a curve can be represented as a function in the polar coordinate system, its essential geometric characteristics can be systematically determined using the proposed analytical algorithm. This approach enables the construction and analysis of the curve without relying exclusively on computational software.

6 Discussion

The obtained results demonstrate that the analysis of curves defined in polar coordinates requires a broader approach than the standard investigation of functions in the Cartesian coordinate system. In Cartesian coordinates, function analysis generally follows a well-established sequence involving the domain, extrema, monotonicity, convexity, inflection points, and asymptotes. However, for curves represented in polar coordinates, the behavior of the radius function and its relationship with the angular variable must also be taken into account.

An important feature of the proposed approach is the combined use of polar and Cartesian coordinate systems. The polar representation makes it possible to investigate the behavior of the radius function, determine its domain and characteristic angular values, and identify asymptotic behavior. The transformation to Cartesian coordinates provides additional information about the geometric position of extremum points and facilitates the interpretation of the resulting curve.

The results obtained in the investigated example also demonstrate the importance of separately considering the components of a curve when its equation leads to more than one radial function. As shown in the calculations, different branches may have different domains, extremum properties, and asymptotic behavior. Therefore, analyzing the branches separately and subsequently combining them into the complete graph provides a more systematic representation of the original curve. This feature is particularly important for curves containing discontinuities or unbounded branches.

Particular attention should be paid to the determination of asymptotes. In the investigated example, the limiting behavior of the radius function was analyzed at the boundary points of its domain, after which the corresponding oblique asymptotes were obtained analytically. Their inclusion in the final graph provides a more complete representation of the behavior of the curve at large distances from the pole.

Another advantage of the proposed algorithm is that the principal characteristics of the curve can be obtained analytically without relying on specialized computational software. Computer programs can significantly simplify graphical visualization; however, analytical determination of the domain, extrema, convexity or concavity, and asymptotes provides a clearer understanding of the mathematical structure of the curve and the relationships between its geometric characteristics.

From a methodological perspective, the proposed sequence of analysis can also be useful in teaching higher mathematics. Instead of obtaining only a computer-generated graph, students can follow the mathematical process through which its characteristic elements are determined. This contributes to a deeper understanding of derivatives, extrema, limits, coordinate transformations, and asymptotic behavior within a single mathematical problem.

At the same time, the present study is based on the detailed analysis of a particular type of curve. Therefore, the proposed algorithm should not be interpreted as a universal procedure applicable without modification to every curve represented in polar coordinates. Its applicability depends on the mathematical form of the radial function, its domain, continuity, and asymptotic behavior. Further investigation of other classes of polar curves would make it possible to evaluate the broader applicability of the proposed approach.

7 Conclusion

This study developed and demonstrated a systematic algorithm for the comprehensive analysis of a curve with asymptotes defined in the polar coordinate system. The proposed approach combines analytical procedures in polar and Cartesian coordinates and provides a sequential framework for identifying the principal geometric characteristics of a curve.

The analysis showed that the investigation should include determining the domain of the radial function, identifying its critical and extremum points, examining convexity and concavity, transforming characteristic points into Cartesian coordinates, and determining the corresponding asymptotes. Particular attention was given to oblique asymptotes, which represent an important characteristic of the behavior of unbounded curves in the polar coordinate system.

The application of the algorithm to the investigated curve demonstrated that different branches of the same curve may exhibit different domains, extremum properties, and asymptotic behavior. Their separate analysis and subsequent graphical combination make it possible to obtain a more complete representation of the original curve.

The proposed analytical approach allows the essential characteristics of a curve to be determined without exclusive reliance on computational software. Therefore, it has methodological value for teaching higher mathematics, as it enables students to understand the mathematical principles underlying the construction of a curve rather than relying solely on its computer visualization.

The results also indicate potential practical applicability of the proposed approach to problems in which geometric trajectories and unbounded curves are analyzed, including problems in physics, engineering, and mathematical modeling. Further research may focus on applying and adapting the proposed algorithm to other classes of curves defined in polar coordinates and comparing analytical results with computer-based methods of curve analysis.

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